Perron: Über geodätische rhombische Netze auf krummen Flächen

Math. Zeit. 24 (1925). p. 170.

Liouville surface.

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

ニ於テ $E = G$ ナルトキ, parametric ~~lines~~ curves ハ rhombic system ヲナストイフ. 今コレガ geodesic ナルタメノ condition ヲ求メテミル.

一般ニ geodesic line ノ dif. eq. ハ

$$\frac{\partial}{\partial v}\sqrt{E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2} = \frac{d}{du}\left(\frac{F + G\frac{dv}{du}}{\sqrt{E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2}}\right).$$

$v = \text{const}$ ガ geod. ナルタメニハ $\frac{dv}{du} = 0.$

$$\frac{\partial}{\partial v}\sqrt{E} = \frac{\partial}{\partial u}\left(\frac{F}{\sqrt{E}}\right).$$

同様ニ $u = \text{const.}$ ガ geod. ナルタメニハ $\frac{\partial}{\partial u}\sqrt{G} = \frac{\partial}{\partial v}\left(\frac{F}{\sqrt{G}}\right).$

コレニ $E = G$ ナレバ

$$\frac{\partial}{\partial v}\sqrt{E} = \frac{\partial}{\partial u}\left(\frac{F}{\sqrt{E}}\right), \qquad \frac{\partial}{\partial u}\sqrt{E} = \frac{\partial}{\partial v}\left(\frac{F}{\sqrt{E}}\right).$$

$$\therefore \quad \frac{\partial^2}{\partial v^2}\sqrt{E} = \frac{\partial^2}{\partial u^2}\sqrt{E}, \qquad \frac{\partial^2}{\partial u^2}\left(\frac{F}{\sqrt{E}}\right) = \frac{\partial^2}{\partial v^2}\left(\frac{F}{\sqrt{E}}\right).$$

general integral ハ

$$\sqrt{E} = f(u+v) + \varphi(u-v), \qquad \frac{F}{\sqrt{E}} = f_1(u+v) + \varphi_1(u-v).$$

$$\left\{\begin{aligned} f' - \varphi' &= f_1' + \varphi_1', \\ f' + \varphi' &= f_1' - \varphi_1'. \end{aligned}\right. \qquad \begin{aligned} f' &= f_1' \\ \varphi' &= -\varphi_1' \end{aligned} \qquad \left.\begin{aligned} f_1 &= f + c_1, \\ \varphi_1 &= -\varphi + c_2. \end{aligned}\right)$$

$$\frac{F}{\sqrt{E}} = f(u+v) - \varphi(u-v) + 2c.$$

$$(1)\quad \left\{\begin{aligned} u + v &= \lambda, \\ u - v &= \mu \end{aligned}\right. \qquad E = \left[f(\lambda) + \varphi(\mu)\right]^2 = G$$

$$F = f^2(\lambda) - \varphi^2(\mu) + 2c\left[f(\lambda) + \varphi(\mu)\right].$$

$$ds^2 = \left[f(\lambda) + \varphi(\mu)\right]^2\left[\left(\frac{d\lambda + d\mu}{2}\right)^2 + \left(\frac{d\lambda - d\mu}{2}\right)^2\right]$$

$$+ 2\left\{f^2(\lambda) - \varphi^2(\mu) + 2c\left[f(\lambda) + \varphi(\mu)\right]\right\}\frac{d\lambda + d\mu}{2}\cdot\frac{d\lambda - d\mu}{2}$$

$$ds^2 = [f(\lambda)+\varphi(\mu)]\{[f(\lambda)+c]\,d\lambda^2 + [\varphi(\mu)-c]\,d\mu^2\}.$$

$$\int\sqrt{f(\lambda)+c}\;d\lambda = \xi, \qquad \int\sqrt{\varphi(\mu)-c}\;d\mu = \eta$$

トオケバ

$f(\lambda) = F(\xi)$, $\qquad ds^2 = [F(\xi)+\Phi(\eta)](d\xi^2+d\eta^2).$

Liouville surface

逆ニ、Liouville surface ニ於テハ

$\sqrt{F(\xi)+c}\;d\lambda = d\xi \qquad \lambda = \int\frac{d\xi}{\sqrt{F(\xi)+c}}$

$$\mu = \int\frac{d\eta}{\sqrt{F(\eta)-c}}$$

トオケ (c ハ arbitrary const.) 。ココニ

$F(\xi) = f(\lambda),$

$F(\eta)-c = \varphi(\mu)$

$d\xi = \sqrt{F(\xi)}\;d\lambda = [f(\lambda)+c]\,d\lambda.$

$\sqrt{f(\lambda)+c}\;d\lambda = d\xi \qquad f(\lambda) = F(\xi).$

$\sqrt{F(\xi)+c}\;d\lambda = d\xi$

$$d\lambda = \frac{d\xi}{\sqrt{F(\xi)+c}}.$$

$$(f+\varphi)^2 \frac{d\lambda^2}{2} + (f+\varphi)^2 \frac{du^2}{} -$$

$$+2\left\{f^2-\varphi^2+2c(f+\varphi)\right\}\frac{d\lambda^2-du^2}{4}$$

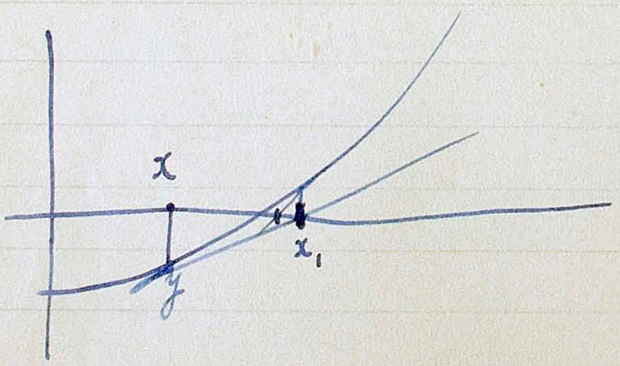
$$= \frac{(f+\varphi)^2 + \left[f^2-\varphi^2+2c(f+\varphi)\right]}{2} d\lambda^2$$

$$+ \frac{(f+\varphi)^2 - \left[f^2-\varphi^2+2c(f+\varphi)\right]}{2} du^2$$

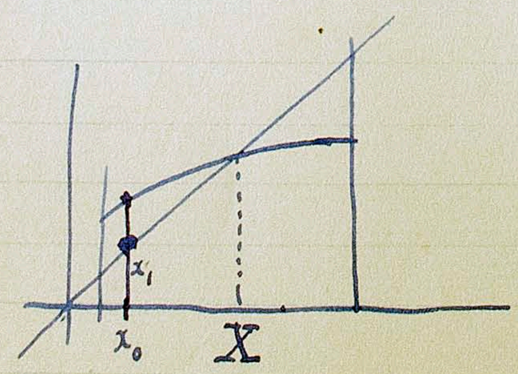
$$f^2+2f\varphi+\varphi^2$$
$$+ f^2-\varphi^2+2c(f+\varphi)$$
$$2f^2+2f\varphi+2c(f+\varphi)$$
$$=2f(f+\varphi)+2c(f+\varphi)$$
$$=2(f+\varphi)(f+c)$$

$$2f\varphi+2\varphi^2-2c(f+\varphi)$$
$$2(f+\varphi)(\varphi-c)$$

$$ds^2=$$



$$Y-y=f'(x)(X-x).$$
$$-y=f'(x)(x_1-x).$$
$$x_1=x-\frac{y}{f'(x)}$$



$$ds^2 = [f(\lambda)+\varphi(\mu)]\{[f(\lambda)+c]\,d\lambda^2 + [\varphi(\mu)-c]\,d\mu^2\}. \qquad (2)$$

$$\int\sqrt{f(\lambda)+c}\,d\lambda = \xi, \qquad \int\sqrt{\varphi(\mu)-c}\,d\mu = \eta,$$

$$f(\lambda) = F(\xi), \qquad \varphi(\mu) = \Phi(\eta)$$

トオケバ:

$$ds^2 = [F(\xi)+\Phi(\eta)](d\xi^2+d\eta^2). \qquad (3)$$

Liouville surface.

逆ニ Liouville surface $\cancel{(3)}$ ガ与ヘラレタトキハ,

$$\lambda = \int\frac{d\xi}{\sqrt{F(\xi)+c}} \qquad \mu = \int\frac{d\eta}{\sqrt{\Phi(\eta)-c}}$$

トセ,

$$F(\xi) = f(\lambda), \qquad \Phi(\eta) = \varphi(\mu)$$

トオケバ;

$$\sqrt{f(\lambda)+c}\,d\lambda = d\xi, \qquad \sqrt{\varphi(\mu)-c}\,d\mu = d\eta.$$

トオケバ (2) トナル.

由テ (1) ヨリ $\lambda+\mu = \text{const}$ 又ハ $u = \text{const}$, $\lambda-\mu = \text{const}$ 又ハ $v = \text{const}$

ハ geodesic rhombic system ヲナス.

コレハ ∞^1 system アリ

$$\left.\begin{aligned}\int\frac{d\xi}{\sqrt{F(\xi)+c}} + \int\frac{d\eta}{\sqrt{\Phi(\eta)-c}} = \text{const},\\ \int\frac{d\xi}{\sqrt{F(\xi)+c}} - \int\frac{d\eta}{\sqrt{\Phi(\eta)-c}} = \text{const}.\end{aligned}\right\}$$